## Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis I

Final examination Date : Nov. 6, 2024 Total Marks: 105 Time: 3 hours Maximum marks: 100 Instructor: B V Rajarama Bhat

(1) Let S be the set of integer sequences converging to 5:

$$S = \{\{x_n\}_{n \in \mathbb{N}} : x_n \in \mathbb{Z}, \lim_{n \to \infty} x_n = 5\}.$$

Is S countable or uncountable? Prove your claim. [15]

- (2) Show that a sequence of real numbers is Cauchy if and only if it is convergent. [15]
- (3) Find lim sup and lim inf of following sequences of real numbers (Do provide proofs of your claims):
  - (i)  $\{a_n\}_{n\geq 1}$  where  $a_n = \frac{1}{n+2} \frac{1}{n}$  for  $n \in \mathbb{N}$ ; (ii)  $\{b_n\}_{n>1}$  where

(11) 
$$\{b_n\}_{n\geq 1}$$
 when

$$b_n = \begin{cases} 2 & \text{if } n = 3k - 2, k \in \mathbb{N} \\ 5 + \frac{10}{n} & \text{if } n = 3k - 1, k \in \mathbb{N} \\ (-1)^n 4 & \text{if } n = 3k, k \in \mathbb{N} \end{cases}$$

[15]

- (4) Let  $f, g : \mathbb{R} \to \mathbb{R}$  be continuous functions. Show that  $h : \mathbb{R} \to \mathbb{R}$ defined by  $h(x) = \min\{f(x), g(x)\}\$  is a continuous function. Show that if f, g are differentiable at c and  $f(c) \neq g(c)$  then h is differentiable at |15|c.
- (5) Let  $g: [0,1] \to \mathbb{R}$  be a function such that g is strictly increasing and it satisfies intermediate value property, that is, for any y with g(0) < y < g(1), there exists 0 < x < 1 such that g(x) = y. Show that q is continuous. |15|
- (6) State and prove Rolle's theorem.

- [15]
- (7) Let  $u: [0,2] \to \mathbb{R}$  be a differentiable function such that u(0) = 1 and  $u'(x) \leq 2$  for all  $x \in [0,2]$ . Show that the maximum possible value of u(2) is 5. Obtain one such function with u(2) = 5.  $\left[15\right]$

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Chain rule